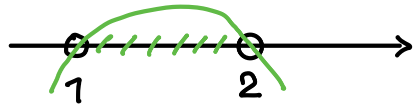
Lösningsskiss till tentamen i Matematisk analys 764G07/TEN1.

2021-08-16

1a $f(x) = \ln \frac{1-x}{x-2}$. $D_f = \{x \in \mathbb{R} : \frac{1-x}{x-2} > 0\}$

$\frac{1-x}{x-2} > 0 \iff (1-x)(x-2) > 0$



Svar: $D_f = \{x \in \mathbb{R} : 1 < x < 2\}$.

1b $8^x - 2^{2x+1} - 2^x + 2 = 0 \iff (2^x)^3 - 2\cdot(2^x)^2 - 2^x + 2 = 0$

$\Big/ \begin{matrix} t = 2^x \\ t > 0 \end{matrix} \Big/ \iff \begin{cases} t^3 - 2t^2 - t + 2 = 0 \\ t > 0 \end{cases} \iff t^2(t-2) - (t-2) = 0$

$\iff (t^2-1)(t-2) = 0 \iff t_1 = 1,\ t_2 = 2,\ t_3 = -1$

falsk rot ty $t_3 < 0$

$\Rightarrow 2^x = 1$ eller $2^x = 2 \Rightarrow x_1 = 0,\ x_2 = 1$.

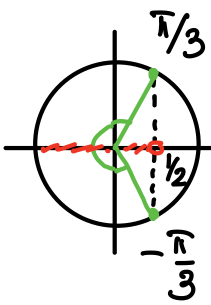
Svar: $x_1 = 0,\ x_2 = 1$.

1c $f(x) = \frac{e^{3x}}{2}$, $f : \mathbb{R} \to \mathbb{R}_+ \Rightarrow y = \frac{e^{3x}}{2} \Rightarrow 2y = e^{3x} \Rightarrow 3x = \ln(2y)$

$\Rightarrow x = \frac{1}{3}\ln(2y),\ y > 0$.

Svar: $f^{-1}(x) = \frac{1}{3}\ln(2x),\ x > 0$

2a



$\cos 4x < \frac{1}{2} \iff \frac{\pi}{3} + 2\pi n < 4x < \frac{5\pi}{3} + 2\pi n,\ n \in \mathbb{Z}$

Svar: $\frac{\pi}{12} + \frac{\pi n}{2} < x < \frac{5\pi}{12} + \frac{\pi n}{2},\ n \in \mathbb{Z}$.

2b $\sin^2 x + 2\cos x + 2 = 0 \ \Big/ \sin^2 x = 1 - \cos^2 x \Big/ \Rightarrow$

$1 - \cos^2 x + 2\cos x + 2 = 0 \Rightarrow \cos^2 x - 2\cos x - 3 = 0$

$\Big/ \begin{matrix} t = \cos x \\ -1 \le t \le 1 \end{matrix} \Big/ \ t^2 - 2t - 3 = 0 \Rightarrow (t+1)(t-3) = 0 \Rightarrow$

$t = -1 \Rightarrow \cos x = -1 \Rightarrow x = \pi + 2\pi n,\ n \in \mathbb{Z}$

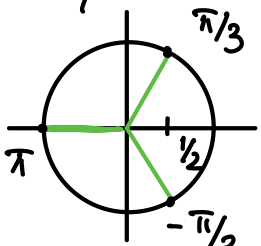
$t = 3$ – falsk rot ty $-1 \le t \le 1$. Svar: $x = \pi + 2\pi n,\ n \in \mathbb{Z}$

3a $z^4 - z^3 + z - 1 = 0 \iff z^3(z-1) + (z-1) = 0 \iff$

$(z^3+1)(z-1) = 0 \Rightarrow z = 1$ eller $z^3 = -1$

$z^3 = -1 \;/\; z = re^{i\theta},\ r>0 \;/ \Rightarrow r^3 e^{i3\theta} = e^{i\pi} \Rightarrow$

$\begin{cases} r^3 = 1 \Rightarrow r = 1 \\ \theta = \dfrac{\pi + 2\pi k}{3}, \quad k = 0,1,2 \end{cases}$



$e^{i\frac{\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = $

$= \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$e^{-i\frac{\pi}{3}} = \cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})$

$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$e^{i\pi} = -1.$

Svar: $z_1 = 1,\ z_{2,3} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2},\ z_4 = -1.$

3b $|z| = \left|\dfrac{(1+i\sqrt{3})^6}{(1-i)^4}\right| = \dfrac{|1+i\sqrt{3}|^6}{|1-i|^4} = \dfrac{\left(\sqrt{1^2+(\sqrt{3})^2}\right)^6}{\left(\sqrt{1^2+(-1)^2}\right)^4} =$

$= \dfrac{2^6}{2^2} = 2^4 = 16$

Svar: $|z| = 16.$

4 $f(x) = \dfrac{e^{3x}}{x+2}$. • $D_f = \{x \in \mathbb{R} : x \neq -2\}$

• $\lim\limits_{x\to-\infty} f(x) = 0 \Rightarrow y = 0$ - vågrät asymptot

$\lim\limits_{x\to-2^-} f(x) = -\infty,\ \lim\limits_{x\to-2^+} f(x) = +\infty \Rightarrow x = -2$ - lodrät asymptot

$\lim\limits_{x\to\infty} f(x) = \infty$

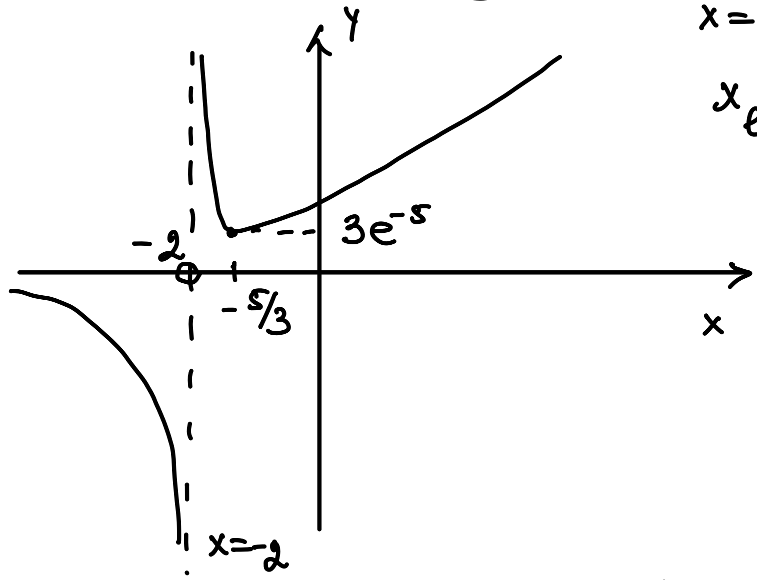
• $f'(x) = \dfrac{3e^{3x}(x+2) - e^{3x}}{(x+2)^2} = \dfrac{e^{3x}(3x+5)}{(x+2)^2} = 0 \iff$

$x = -\frac{5}{3}$

	$x<-2$	-2	$-2<x<-\frac{5}{3}$	$-\frac{5}{3}$	$x>-\frac{5}{3}$
f'	$-$	o	$-$	o	$+$
f	$\searrow$		$\searrow$		$\nearrow$

lok. min $f(-\frac{5}{3}) = \dfrac{e^{-5}}{1/3} = \dfrac{3}{e^5}.$

Svar:



$x=-2$, $y=0$ - asymtoter

$x_{\text{lok. min}} = -\frac{5}{3}$, $f(-5/3) = \frac{3}{e^5}$.

5a $$\lim_{x\to 2} \frac{x^2-x-2}{x-2} = \left[\frac{0}{0}\right] = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

5b $$\lim_{x\to 0} \frac{e^{5x}-1}{\sin 3x} = \lim \frac{e^{5x}-1}{5x}\cdot\frac{5x}{3x}\cdot\frac{3x}{\sin 3x} = \frac{5}{3}.$$

5c $$\lim_{x\to\infty}\left(\sqrt{x^2-5x+4}-\sqrt{x^2-x}\right) = [\infty-\infty] = \lim_{x\to\infty}\frac{(x^2-5x+4)-(x^2-x)}{\sqrt{x^2-5x+4}+\sqrt{x^2-x}} =$$

$$= \lim_{x\to\infty}\frac{-4x+4}{\sqrt{x^2-5x+4}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x\left(-4+4/x\right)}{x\left(\sqrt{1-\frac{5}{x}+\frac{4}{x^2}}+\sqrt{1-\frac{1}{x}}\right)} = \frac{-4}{2}$$

Svar: a) 3 b) $\frac{5}{3}$ c) -2.

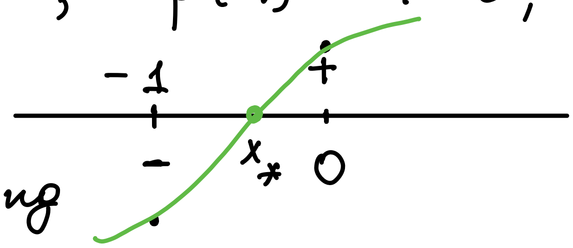
6 $p(x) = 3x^3 - x^2 + x + 1$. $p'(x) = 9x^2 - 2x + 1 =$

$= 9\left(x-\frac{1}{9}\right)^2 - \frac{1}{9} + 1 = 9\left(x-\frac{1}{9}\right)^2 + \frac{8}{9} > 0$ för alla x

$\Rightarrow$ $p(x)$ är strängt växande samt

$\lim_{x\to-\infty} p(x) = -\infty$, $\lim_{x\to+\infty} p(x) = +\infty$, $p(-1) = -4 < 0$,

$p(0) = 1 > 0$, p är kontin. $\Rightarrow$



Svar: $p(x)$ har exakt en lösning

$x_* \in]-1,0[$ enligt satsen om mellanliggande värde

7 $f(x) = \dfrac{x^3}{x^2-1}$. • $D_f = \{ x \in \mathbb{R} : x \neq \pm 1 \}$.

• $\lim\limits_{x \to -\infty} f(x) = -\infty$; $\lim\limits_{x \to -1^-} f(x) = -\infty$, $\lim\limits_{x \to -1^+} f(x) = +\infty$

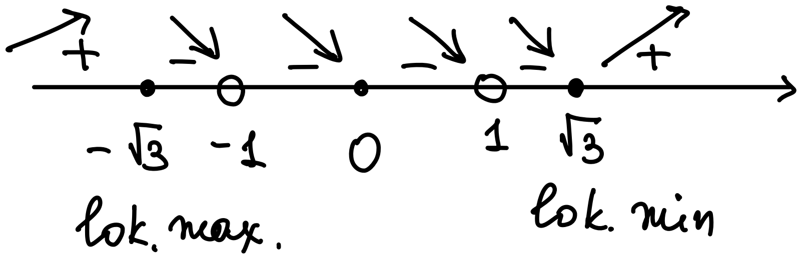
$\Rightarrow$ $x = -1$ - lodrät asymptot.

$\lim\limits_{x \to 1^-} f(x) = -\infty$, $\lim\limits_{x \to 1^+} f(x) = \infty \Rightarrow x = 1$ - lodrät asymptot

$\lim\limits_{x \to \infty} f(x) = \infty$.

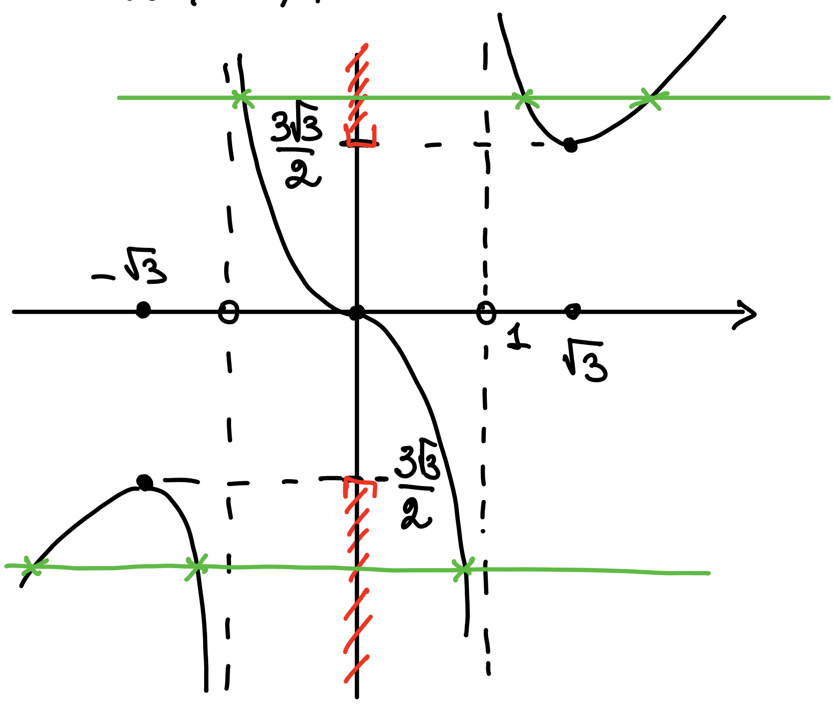
• $$f'(x) = \frac{3x^2(x^2-1) - 2x \cdot x^3}{(x^2-1)^2} = \frac{x^4 - 3x^2}{(x^2-1)^2} = \frac{x^2(x^2-3)}{(x^2-1)^2}$$

$f'(x) = 0 \iff x = 0, x = \pm\sqrt{3}$.



$f_{l.max} = f(-\sqrt{3}) = -\dfrac{3\sqrt{3}}{2}$

$f_{l.min} = f(\sqrt{3}) = \dfrac{3\sqrt{3}}{2}$



Svar: $f(x) = a$ har exakt tre olika reella lösningar för alla a som ligger i $]-\infty, -\frac{3\sqrt{3}}{2}[$ eller i $]\frac{3\sqrt{3}}{2}, \infty[$.